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Lorentz Invariance, Approach to Helicity, and Gauge Transformations for Spin-1

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Spin-1 fields are constructed automatically using the regular representation of the three-dimensional complex orthogonal group and the group's isomorphism with the Lorentz group. The fields and potentials are examined as one lets their mass go to zero. Going to masslessness after the differentiation of potentials results in a consistent formulation of the fields. The behavior of the massive and massless potentials under rotations in the particle frame is examined. The loss of global degrees of freedom as one goes from one to the other is made up by the appearance of gradient terms.

1. INTRODUCTION

Biritz' comprehensive work on massive particle equations (Biritz, 1979) and Weinberg's Brandeis lectures on the theory of massless particles (Weinberg, 1964) strongly influenced this author's formal approach to the age-old problem of the transition from massive to massless particle fields. Of the cases so far considered in some detail, spin-1/2 and spin-1, the latter is briefly described here, being the more complex of the two since it depends not only on the fields proper but also on the potentials. Thus, gauge transformations come up as one goes to the limit of masslessness or, more to the point, as one reaches helicity.

2. NOMENCLATURE

The generators $J_{\nu\mu}(\nu, \mu = 0\text{–}3)$ of L_h, the homogeneous Lorentz group, are divided into the two sets $J_k \equiv J_{lm}$ ($k, l, m = 1$—3, used cyclically) and

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$K_k \equiv J_{k0}$. The vectors

$$\mathbf{A} = \frac{\mathbf{J} + i\mathbf{K}}{2} \qquad \text{and} \qquad \mathbf{B} = \frac{\mathbf{J} - i\mathbf{K}}{2} \tag{1}$$

then independently from one another satisfy the angular momentum commutation relations. The matrix representations of L_h of interest here are denoted by (A, B), A and B being the maximum values of the diagonal A_3 and B_3, respectively. The massive fields ψ_A and ψ_B are associated with (1,0) and (0,1), respectively. So are the fields with helicity, ψ_- and ψ_+. The massive potential Φ is associated with (1/2,1/2). The massless potential can be separated into the helicity carrying parts Φ_- and Φ_+. Owing to the occurrence of the three mentioned representations we have boost matrices called B_A, B_B, and B. It suffices to have one type of rotation matrix, the three-dimensional R. Finally, the unitary operators representing elements of the inhomogeneous Lorentz group, L_i, are called U.

3. THE MASSIVE FIELDS

Under infinitesimal transformations the massive fields behave as follows:

$$\psi'^k_{A,B} = (1 + i\varepsilon_m \mathsf{J}_m)^k{}_l \psi^l_{A,B} \tag{2a}$$

and

$$\psi'^k_{A,B} = (1 \pm \varepsilon_m \mathsf{J}_m)^k{}_l \psi^l_{A,B} \tag{2b}$$

equations (2a) referring to rotations, equations (2b) to Lorentz transformations; the ε_k are real.

In a Cartesian system the representations are (complex) orthogonal and therefore the vertical position of the roman transformation indices becomes irrelevant. Also, since we are dealing with the regular representations of $\mathbf{A}$ and $\mathbf{B}$, the $\psi^k_A = \psi_{Ak}$ transform like the A_k and the $\psi^k_B = \psi_{Bk}$ like the B_k. Consequently, equations (1) allow us to define antisymmetric tensors in space-time:

$$F_A^{\nu\mu} \equiv \begin{cases} -\psi^k_A \\ i\psi^k_A \end{cases} \qquad \text{and} \qquad F_B^{\nu\mu} \equiv \begin{cases} \psi^k_B, & (\nu = l, \mu = m) \\ i\psi^k_B, & (\nu = 0, \mu = k) \end{cases} \tag{3}$$

The overall minus sign in the first set of identities as compared to the relations between $J_{\nu\mu}$ and A_k turns out to be convenient because of the form the $F_{A,B}$ assume in the massless limit.

Our fields may be written in the following form:

$$\psi_{A,B}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E)^{1/2}} \mathsf{R}^{-1}(\theta,\varphi) \mathsf{B}_{A,B}(\alpha)$$

$$\times \frac{1}{\sqrt{2}} \mathsf{P} \left[\begin{pmatrix} a^+(\mathbf{p}) \\ a^0(\mathbf{p}) \\ a^-(\mathbf{p}) \end{pmatrix} e^{ipx} - \begin{pmatrix} a^-(\mathbf{p}) \\ a^0(\mathbf{p}) \\ a^+(\mathbf{p}) \end{pmatrix}^* e^{-ipx} \right] \tag{4}$$

P is a matrix transforming spherical to Cartesian Fourier coefficients. The boosts

$$\mathsf{B}_{A,B} = \begin{pmatrix} \cosh\alpha & \pm i \sinh\alpha & 0 \\ \mp i\sinh\alpha & \cosh\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

relate the Fourier coefficients to the rest frame and R^{-1} projects the resulting components onto the specific coordinate system. The relative minus sign and the choice of one set of Fourier coefficients in terms of the other leads to a real field for $F = (F_A + F_B)/2$.

The reduction of an infinite component base of L_i to a finite one of L_h is possible if the complete set of spin projections, here denoted by superscripts on $a(\mathbf{p})$, is contained in the representation of L_h. For spin-1 and the representations (1,0) and (0,1) that holds practically trivially.

The contributions to the integrand in equation (4) from a particular momentum, say, along z, are

$$\mp \frac{1}{\sqrt{2}} \left[e^{\mp\alpha} a^+(\mathbf{p}) + e^{\pm\alpha} a^-(\mathbf{p}) \right] e^{i(pz-Et)} = f^{23}_{A,B}$$

$$\mp \frac{i}{\sqrt{2}} \left[e^{\mp\alpha} a^+(\mathbf{p}) - e^{\pm\alpha} a^-(\mathbf{p}) \right] e^{i(pz-Et)} = f^{31}_{A,B} \tag{5}$$

and

$$\mp a^0(\mathbf{p}) e^{i(pz-Et)} = f^{12}_{A,B}, \qquad f^{0k}_{A,B} = \mp i f^{lm}_{A,B}$$

For the sake of clarity, we have only written the positive frequency parts. As $\alpha \to \infty$ $(m \to 0)$, f_A approaches negative helicity and f_B positive; those are exactly the helicities of the massless fields: the Fourier integral of the massless field $\psi_-(\psi_+)$ is given by the same expression as for the massive field, equation (4), but without a^+, $a^0(a^-, a^0)$. Furthermore, e^α then equals p/p_r, p_r being a reference momentum.

4. THE MASSIVE POTENTIAL

We write the massive 4-potential, in analogy to equation (4), as the integral

$$\Phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E)^{1/2}} \begin{pmatrix} \mathsf{R}^{-1}(\theta,\varphi) & 0 \\ 0 & 1 \end{pmatrix} \mathsf{B}(\alpha)$$

$$\times \frac{1}{\sqrt{2}} \mathsf{P}_4 \left[\begin{pmatrix} a^+(\mathbf{p}) \\ a^0(\mathbf{p}) \\ a^-(\mathbf{p}) \end{pmatrix} e^{ipx} + \begin{pmatrix} a^-(\mathbf{p}) \\ a^0(\mathbf{p}) \\ a^+(\mathbf{p}) \end{pmatrix}^* e^{-ipx} \right] \tag{6}$$

where B is the boost along z in space-time, and P_4 is the matrix P in equation (4) augmented by a fourth row of zeros. Clearly, Φ satisfies the Lorentz condition.

From equation (6) we extract the particular potential

$$\begin{aligned} \phi^1 &= \frac{1}{\sqrt{2}} (a^+(\mathbf{p}) + a^-(\mathbf{p})) e^{i(pz - Et)} \\ \phi^2 &= \frac{i}{\sqrt{2}} [a^+(\mathbf{p}) - a^-(\mathbf{p})] e^{i(pz - Et)} \\ \phi^3 &= \frac{p^0}{m} a^0(\mathbf{p}) e^{i(pz - Et)} \\ \phi^0 &= \frac{p}{m} a^0(\mathbf{p}) e^{i(pz - Et)} \end{aligned} \tag{7}$$

As the mass approaches zero, the transverse components vanish relative to the longitudinal ones: only the two zero helicities are contained in $(1/2, 1/2)$.

If we now take the curl, we find the following:

$$\frac{1}{m} \left(\frac{\partial \phi^\mu}{\partial x_\nu} - \frac{\partial \phi^\nu}{\partial x_\mu} \right) = \frac{1}{2} (f_A + f_B)^{\nu\mu} \tag{8}$$

The longitudinal term with indices zero and three vanishes relative to the others as we let the mass vanish. The other longitudinal term, indices one and two, is zero in any case.

From equations (5) we gather, e.g.,

$$\frac{1}{2}(f_A + f_B)^{23} = \frac{1}{\sqrt{2}} \sinh\alpha [a^+(\mathbf{p}) - a^-(\mathbf{p})] e^{i(pz-Et)} \tag{9a}$$

$$\rightarrow \frac{1}{\sqrt{2}} \frac{e^\alpha}{2} [a^+(\mathbf{p}) - a^-(\mathbf{p})] e^{ip(z-t)} \tag{9b}$$

Equation (8) itself follows only after identifying $\sinh\alpha$ with p/m, i.e., upon calling on L_i. Equation (8) can, of course, be immediately generalized to relate Φ and F.

5. THE MASSLESS FIELDS AND POTENTIALS

The limiting expression on the right of equation (9b) is, as expected, found in $(F_+ + F_-)/2$ where, like in equations (3), the tensors are given in terms of the complex vectors by

$$F_-^{\nu\mu} \equiv \begin{cases} -\psi_-^k \\ i\psi_-^k \end{cases} \quad \text{and} \quad F_+^{\nu\mu} \equiv \begin{cases} \psi_+^k \\ i\psi^{k+} \end{cases} \tag{10}$$

and the vectors are expressed by

$$\psi_{-,+}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E)^{1/2}} \mathsf{R}^{-1} \mathsf{B}_{A,B} \times \frac{1}{\sqrt{2}} \mathsf{P} \left[\begin{pmatrix} a^+(\mathbf{p})_+ \\ 0 \\ a^-(\mathbf{p})_- \end{pmatrix} e^{ipx} - \begin{pmatrix} a^-(\mathbf{p})_+ \\ 0 \\ a^+(\mathbf{p})_- \end{pmatrix}^* e^{-ipx} \right] \tag{11a}$$

or

$$\psi_{-,+}^k(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{(2E)^{1/2}} e^\alpha e_{-,+}^k(\mathbf{p}) \left[a^{-,+}(\mathbf{p}) e^{ipx} - a^{+,-}(\mathbf{p})^* e^{-ipx} \right] \tag{11b}$$

The last integral contains the polarization vectors (Weinberg, 1964)

$$e^k_{-,+}(\mathbf{p}) = \mathsf{R}^{-lk}{}_l e^l_{-,+}; \qquad e^1_{-,+} = \frac{1}{\sqrt{2}}, \qquad e^2_{-,+} = \mp\frac{i}{\sqrt{2}}, \qquad e^3_{-,+} = 0$$

The massless potentials may conveniently be written in the form

$$\Phi_{-,+}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{d^3p}{(2E)^{1/2}} \begin{pmatrix} \mathsf{R}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \mathsf{B}$$
$$\times \frac{1}{\sqrt{2}} \mathsf{P}_4 \left[\begin{pmatrix} a^+(\mathbf{p})_+ \\ 0 \\ a^-(\mathbf{p})_- \end{pmatrix} e^{ipx} + \begin{pmatrix} a^-(\mathbf{p})_+ \\ 0 \\ a^+(\mathbf{p})_- \end{pmatrix}^* e^{-ipx} \right] \tag{12}$$

Their curls and the fields are related by

$$\frac{1}{p_r}\left(\frac{\partial \Phi^\mu_{-,+}}{\partial x_\nu} - \frac{\partial \phi^\nu_{-,+}}{\partial x_\mu} \right) = F^{\nu\mu}_{-,+} \tag{13}$$

The separability of the massless curl equations clearly allows the choices

$$\Phi = \Phi_+ + \Phi_+ \qquad \text{and} \qquad \Phi = i(\Phi_+ - \Phi_-) \tag{14}$$

The relation

$$2F^{\nu\mu} = \frac{1}{p_r}\left(\frac{\partial \Phi^\mu}{\partial x_\nu} - \frac{\partial \Phi^\nu}{\partial x_\mu} \right) \tag{15}$$

then makes the symmetry under the exchange

$$F^{kl} \to F^{0m}, \qquad F^{0k} \to -F^{lm} \tag{16}$$

quite apparent; see also equations (5) and (7). We are adhering to our convention [see equation (8)] and define for the massless case $F = (F_+ + F_-)/2$.

A unified treatment of the massive and massless curl relations, [equations (8) and (13)], is obtained by writing for a particular $\mathbf{p}$, $p \equiv |\mathbf{p}|$

$$f^{\nu\mu} = \frac{e^\alpha}{p+E}\left(\frac{\partial \phi^\mu}{\partial x_\nu} - \frac{\partial \phi^\nu}{\partial x_\mu} \right) \tag{17}$$

with $f \equiv (1/2)(f_A + f_B)$ or $\equiv (1/2)(f_+ + f_+)$, and $\phi \equiv \phi_- + \phi_+$ in the massless case. Any massless limit must be taken after the differentiation.

6. LORENTZ INVARIANCE AND GAUGE TRANSFORMATIONS

While the absence of longitudinal components from the massless potentials does not bother us as far as the fields are concerned—they do not appear there—it might be worrisome in potential-dependent terms like interactions.

Weinberg (1964) has shown that the apparent lack of Lorentz invariance actually is beneficial rather than a death blow to the theory. Thus, transforming to a different observer results in a Lorentz transformation of the massless (transverse) potential and a gauge transformation. Here we should like to view this state of affairs in the context of the massiveness to masslessness transition.

The product of rotation and boost constitutes a set of wave functions for a particular momentum. Let us call the matrix element of this set $u^{\nu}{}_{\sigma}(\mathbf{p})$. Since we are already in Cartesian coordinates—P has operated on the vector $a^{s}(\mathbf{p})$ $(s = +, 0, -)$— the wave functions are generally not orthogonal for different σ. The covariance of equation (6) is made possible by the fact that the spin implicit in the coefficients $a^{s}(\mathbf{p})$ is also contained in the representation (1/2,1/2).

We have the relation

$$[E(\mathbf{p})]^{1/2}\lambda^{\nu}{}_{\mu}u^{\mu}{}_{k}(\mathbf{p})a^{k}(\mathbf{p}) = [E(\lambda\mathbf{p})]^{1/2}u^{\nu}{}_{k}(\lambda\mathbf{p})U^{-1}(\lambda)a^{k}(\lambda\mathbf{p})U(\lambda) \tag{18}$$

Of course, $|\mathbf{p}|^2 - E^2 = -m^2$.

The $U(\lambda)$ are the unitary transformation operators forced upon us by the use of the $a(\mathbf{p})$ as free particle destruction operators.

Let us consider an arbitrary transformation λ in space-time. Equation (18) becomes

$$\lambda^{\nu}{}_{\mu}u^{\mu}{}_{k}(\mathbf{p}) = u^{\nu}{}_{l}(\lambda\mathbf{p})\mathsf{R}^{l}{}_{k}(\lambda,\mathbf{p}) \tag{19}$$

Since $\mathsf{R}(\lambda, p)$ is given by

$$R^{k}{}_{l}(\lambda,\mathbf{p}) = \mathsf{H}^{-1k}{}_{\nu}(\lambda\mathbf{p})\lambda^{\nu}{}_{\mu}\mathsf{H}^{\mu}{}_{l}(\mathbf{p})$$

and

$$u(\mathbf{p}) = \mathsf{H}(\mathbf{p})$$

(remember, owing to our choice of coordinates $u^{\nu}{}_{\sigma}(0)=\delta^{\nu}{}_{\sigma}$) equation (19) is manifestly satisfied. The matrix H is the general, helicity-preserving boost from rest to momentum **p**.

Equation (19) explicitly guarantees the covariance of the potential Φ^{ν} as given by equation (6).

With $\lambda=\mathsf{HRH}^{-1}$ the transformation λ expresses the observer's view of a rotation of the massive particle's fixed axes, and with

$$\mathsf{H}=\mathsf{B}$$

the momentum **p** is directed along the z axis.

Now let us similarly consider the massless potential. To emphasize the essential points we shall immediately use

$$\lambda=\mathsf{BRB}^{-1} \qquad \text{and } \mathbf{p}=\hat{k}p \qquad (\text{i.e., } \mathbf{p} \text{ along } z) \tag{20}$$

Furthermore, let us first look at a (negative) rotation about the y axis:

$$\lambda=\mathsf{B}(\alpha)\mathsf{R}(-\theta)\mathsf{B}^{-1}(\alpha) \tag{21}$$

where α and θ are the usual parameters; thus, $\tanh\alpha=v$. If now, as before in the massive case, the rotation occurs in the particle's frame, the right side of equation (21) goes over into the matrix

$$\lambda=\begin{pmatrix} 1 & 0 & Y & -Y \\ 0 & 1 & 0 & 0 \\ -Y & 0 & 1-\dfrac{Y^2}{2} & \dfrac{Y^2}{2} \\ -Y & 0 & -\dfrac{Y^2}{2} & 1+\dfrac{Y^2}{2} \end{pmatrix} \tag{22}$$

Where $Y=\lim_{\theta\to 0,\,\alpha\to\infty}\theta e^{\alpha}$.

The generator of this λ is clearly $K_1-J_2\equiv L_1$; see equation (1). The same generator can also be obtained by first sandwiching an infinitesimal rotation between finite boosts, yielding the matrix

$$\lambda=\begin{pmatrix} 1 & 0 & \varepsilon\cosh\alpha & -\varepsilon\sinh\alpha \\ 0 & 1 & 0 & 0 \\ -\varepsilon\cosh\alpha & 0 & 1 & 0 \\ -\varepsilon\sinh\alpha & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

writing this as

$$\lambda=1+i(\varepsilon\sinh\alpha K_1-\varepsilon\cosh\alpha J_2)$$

and then letting the coefficients of K_1 and J_2 approach one another while remaining infinitesimal. The finite transformation subsequently generated by the usual exponentiation is a series that terminates after the second nontrivial term. It is identical to equation (22).

Rotating about the particle fixed x axis results in the observer frame generator $K_2 + J_1 \equiv L_2$; rotations about z clearly do not depend on any relative motion along z.

Thus, infinitesimal rotations in the particle's frame perpendicular to its velocity appear finite to the observer. The relevant body fixed generating transformations are second-order infinitesimals.

The ring made of L_1, L_2, and J_3 generates the well-known Euclidian group in two dimensions, E(2), here relevant to massless particle fields, and resulting from a limiting process applied to the three-dimensional rotation group, which is relevant to massive particles, in particular particles at rest.

We are now in a position to formulate the equivalents of equations (18) and (19) in the massless case. Thus, on one side, there is the transformation

$$U^{-1}(\lambda)a^h(\lambda\mathbf{p})U(\lambda) = \left[\frac{E(\mathbf{p})}{E(\lambda\mathbf{p})}\right]^{1/2} e^{\mp i\varphi}a^h(\mathbf{p}) \tag{24}$$

the superscript h standing for helicity; here $h = \pm 1$. The appearance of only (two) diagonal terms $e^{\mp i\varphi}$ instead of an entire matrix R, as in equation (19), is due to the helicity implicit in the massless operators $a^{\pm}(\mathbf{p})$ (Weinberg, 1964).

On the other side we have to consider

$$\lambda^{\nu}{}_{\mu}u^{\mu}{}_{k}(\mathbf{p})e^{k}_{\pm}\,a^h(\mathbf{p}) \tag{25}$$

$e^k_{\pm}$ are the elements of the polarization tensor P; see eq. (11b). [In the massless case it is most convenient to use the operators $a^h(\mathbf{p})$, as already indicated by the diagonal form of equation (24).] Under the conditions of equations (20), using equation (22), the analogous form involving a (positive) rotation χ about x

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & X & -X \\ 0 & -X & 1-\dfrac{X^2}{2} & \dfrac{X^2}{2} \\ 0 & -X & -\dfrac{X^2}{2} & 1+\dfrac{X^2}{2} \end{pmatrix} \tag{26}$$

where $X = \lim_{\chi \to 0, \alpha \to \infty} \chi e^{\alpha}$, and a rotation about the velocity axis, z, by the angle φ we may then write

$$\lambda^{-1\nu}{}_{k} e^{k}_{\pm} a^{h}(\mathbf{p}) - \frac{1}{\sqrt{2}}(Y \pm iX) a^{h}(\mathbf{p}) \frac{p^{\nu}}{p} = e^{\nu}_{\pm} U(\lambda) a^{h}(\mathbf{p}) U^{-1}(\lambda), \qquad e^{0}_{\pm} = 0$$

and thus finally

$$e^{\nu}_{\pm} a^{h}(\mathbf{p}) - \frac{1}{\sqrt{2}}(Y \pm iX) a^{h}(\mathbf{p}) \frac{p^{\nu}}{p} = \lambda^{\nu}{}_{\sigma} e^{\sigma}_{\pm} U(\lambda) a^{h}(\mathbf{p}) U^{-1}(\lambda) \qquad (27)$$

λ may now contain all three parameters.

Although derived under the assumption that $|\mathbf{p}|$ equals p_3 [see equations (20)], equation (27) is, on account of its covariance, immediately found to be valid for any $\mathbf{p}$. The $e^{\sigma}_{\pm}$ then become

$$e^{k}_{\pm}(\mathbf{p}) = \mathsf{R}^{-1k}{}_{l} e^{l}_{\pm}, \qquad e^{0}_{\pm} = 0$$

and λ is a transformation in the group E(2) which leaves $\mathbf{p}$, E invariant (rather than $p_3 = |\mathbf{p}|$, $E = p_3$).

Equation (27) states that a rotation in "the frame of a massless spin-one particle" perpendicular to its velocity appears to the accordingly transformed observer as a (pure) gauge transformation of the potential of the particle field. Thus, the two global degrees of freedom lost when reaching helicity are regained in the form of gradients.

Clearly, perpendicular rotations can be performed on every Fourier coefficient in equation (12), resulting in a pure gauge transformation of the general potentials $\Phi_{+,-}(x)$.

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